orthogonal basis functions for kinematic modelling

a technical report

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overview

complete set of bi-orthonormal basis functions

$$F_{nmk}(x,y,v) = S_n(R)\,\mathrm{e}^{im\phi}\,\ell_k(v|R)$$

with (x,y) sky position and v line-of-sight velocity and

$$\delta_{nn'}\delta_{mm'}\delta_{kk'} = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,F_{nmk}(x,y,v)\,\bar{F}_{n'm'k'}(x,y,v)$$

$$\delta(x-x')\,\delta(y-y')\,\delta(v-v') = \sum_{nmk} F_{nmk}(x,y,v)\,\bar{F}_{nmk}(x',y',v')$$

allow to expand

$$F(x,y,v) = \sum_{nmk} C_{nmk}\,S_n(R)\,\mathrm{e}^{im\phi}\,\ell_k(v|R)$$

with (e.g. for an *N*-body model)

$$C_{nmk} = \sum_i m_i\,\bar{F}_{nmk}(x_i,y_i,v_{zi})$$

usage in dynamical modelling of discrete data

1. log-likelihood approach:

   modify model to maximise $\mathcal{L} = \sum_j \ln p_j$

   with

$$p_j = F(x_j, y_j, v) *_v \exp\left(-\frac{(v - v_j)^2}{2\sigma_j^2}\right)$$

   => requires velocity error convolution ($*_v$)

2. generalised moment fitting:

   modify model to minimise $\chi^2 = \sum_{nmk} \left[\frac{C_{nmk,\mathrm{model}} - C_{nmk,\mathrm{data}}}{\sigma_{nmk}}\right]^2$

   with

$$C_{nmk,\mathrm{data}} = \sum_j \bar{F}_{nmk}(x_j, y_j, v) \div_v \exp\left(-\frac{(v - v_j)^2}{2\sigma_j^2}\right)$$

   => requires velocity error de-convolution ($\div_v$)

radial basis functions: Sersiclets

Sersic (1963) profile:

$$I_s(R) = \frac{s(n_s)}{2\pi R_0^2}\, \mathrm{e}^{-b[R/R_0]^{1/n_s}}$$

ansatz for radial basis functions:

$$S_n(R) = \bar{S}_n(R) = I_s(R)\, P_n(\xi) \quad \text{with} \quad \xi = 2b[R/R_0]^{1/n_s}$$

gives the bi-orthogonality relation

$$\delta_{nn'} \propto \int_0^\infty \mathrm{d}\xi\, \xi^\alpha\, \mathrm{e}^{-\xi}\, P_n(\xi)\, P_{n'}(\xi) \quad \text{with} \quad \alpha = 2n_s - 1$$

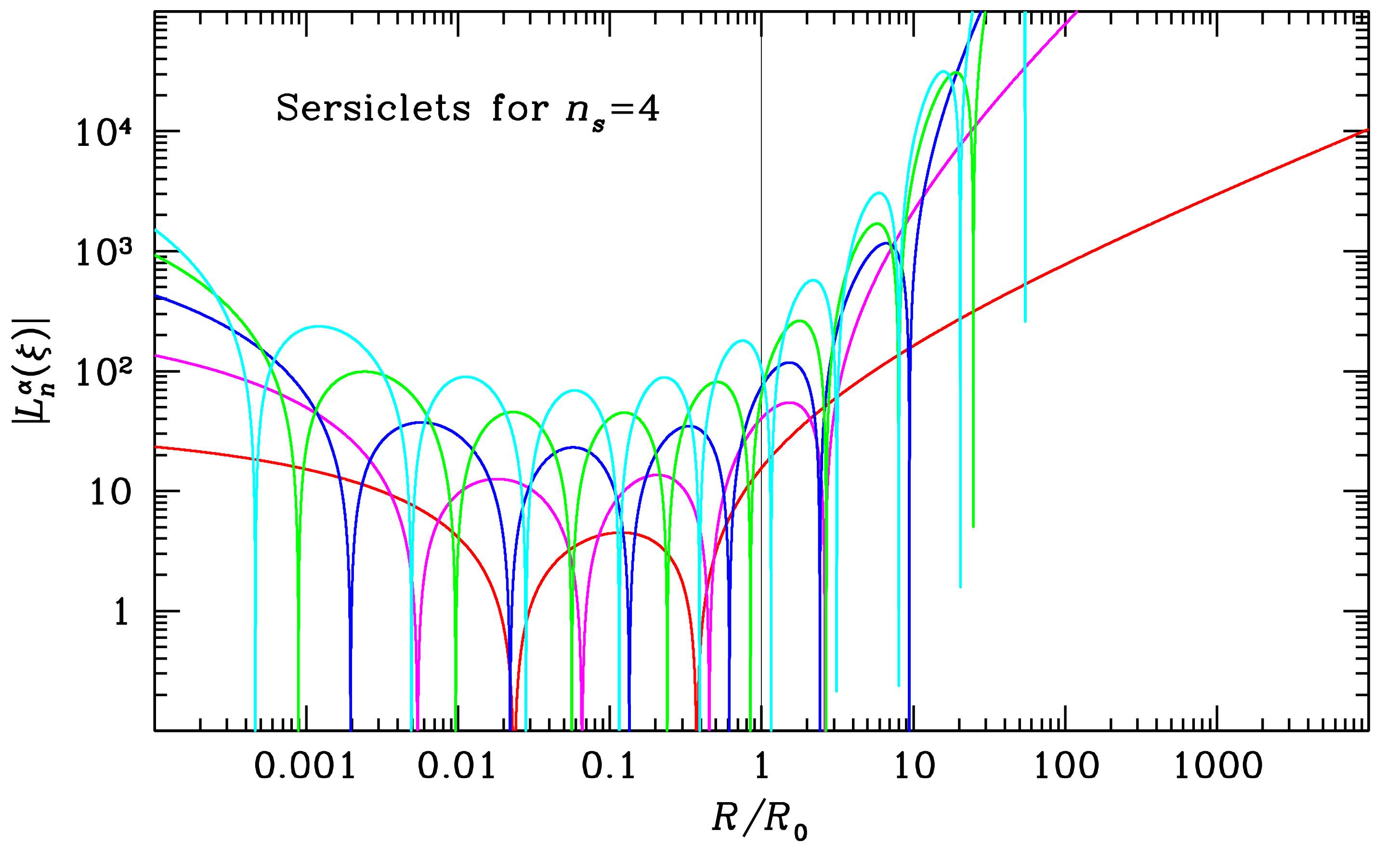
of the Laguerre polynomials

$$L_n^\alpha(x) = \frac{1}{n!}\mathrm{e}^x x^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x} x^{n+\alpha})$$

=> Sersiclets of Andrae, Melchior & Jahnke (2011)

$$S_n(R) = \bar{S}_n(R) \propto I_s(R)\, L_n^{2n_s-1}(\xi)$$

radial basis functions: Sersiclets



problem:

too much resolution at $R < R_0$

=> Andrae et al.: more complicated model for lowest order

=> requires fully numerical treatment

alternatively:

1 generalise $$I_\gamma(R) = \frac{s(n_s,\gamma)}{2\pi R_0^2}\left(\frac{R}{R_0}\right)^{-\gamma} \mathrm{e}^{-b[R/R_0]^{1/n_s}}$$

2 allow $S_n(R) \neq \bar{S}_n(R)$

radial basis functions: beyond Sersiclets

$$I_\gamma(R) = \frac{s(n_s,\gamma)}{2\pi R_0^2} \left(\frac{R}{R_0}\right)^{-\gamma} \mathrm{e}^{-b[R/R_0]^{1/n_s}}$$

the ansatz:

$S_n(R) = I_\gamma(R)\, P_n(\xi)$ and $\bar{S}_n(R) = \xi^\beta\, \mathrm{e}^{(1-c)\xi}\, S_n(R)$

with $\gamma > 0$ and $2(1-\gamma)n_s + \beta > 0$

again leads to Laguerre polynomials:

$$S_n(R) \propto I_\gamma(R)\, L_n^{2(1-\gamma)n_s+\beta-1}(c\xi)$$

=> more general lowest order (parameters R_0, n_s, γ)

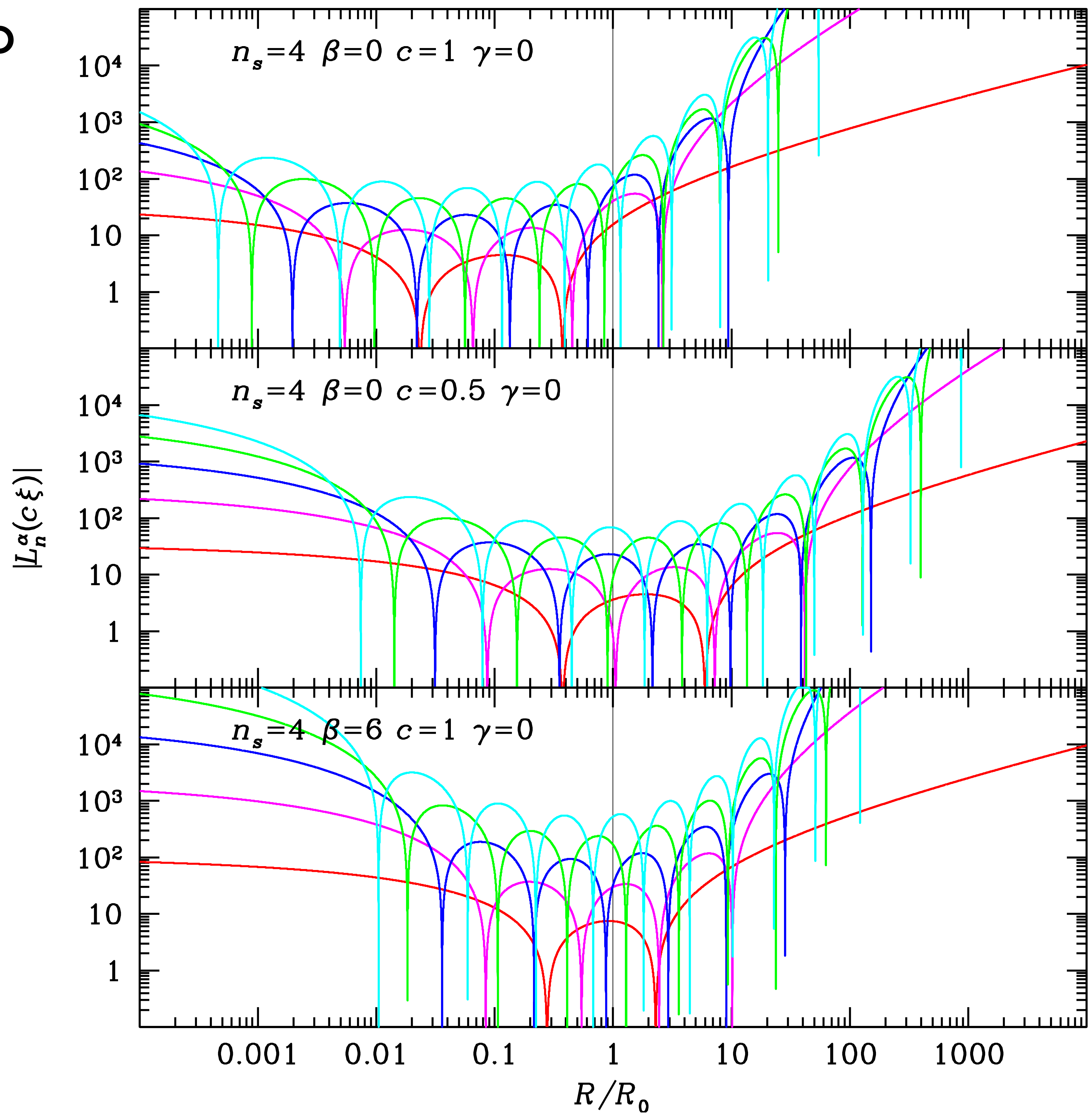
=> more flexible higher order (parameters c, β)

=> still fully analytical (recursive computation etc.)

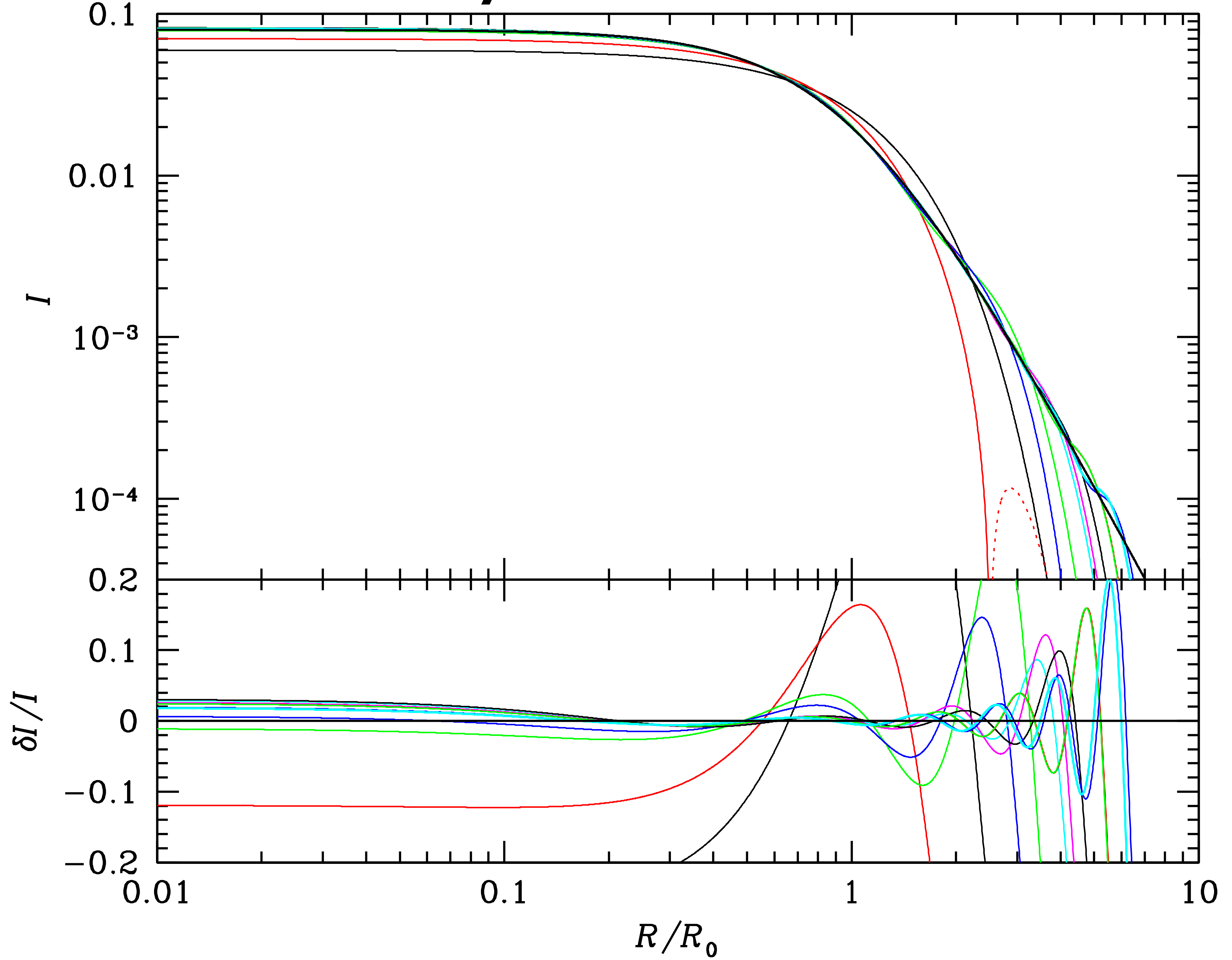
=> useful if truncation at large radii as in Sersic profile

radial basis functions: beyond Sersiclets

new parameters allow to adapt resolution



radial basis functions: beyond Sersiclets



fit of Plummer with Sersiclets ($n_s = 0.6, c = 1, \beta = 0, \gamma = 0$)

radial basis functions: double power law models

double power law model:

$$I_p(R) \propto \left(\frac{R}{R_0}\right)^{-\gamma_0} \left(\left[\frac{R}{R_0}\right]^\eta + 1\right)^{(\gamma_0-\gamma_\infty)/\eta}$$

ansatz:

$$S_n(R) = I_p(R)\, P_n(\xi) \quad \text{and} \quad \bar{S}_n = Z(\xi)\, S_n(R)$$

with

$$\xi \equiv \frac{R^\eta - R_0^\eta}{R^\eta + R_0^\eta}$$

gives the bi-orthogonality relation

$$\delta_{nn'} \propto \int_{-1}^{1} \mathrm{d}\xi\, (1+\xi)^{\frac{2(1-\gamma_0)}{\eta}-1} (1-\xi)^{\frac{2(\gamma_\infty-1)}{\eta}-1} Z(\xi) P_n(\xi) P_{n'}(\xi)$$

For

$$Z(\xi) = (1-\xi)^{\alpha_0}(1+\xi)^{\beta_0} = 2^{\alpha_0+\beta_0}\left(\frac{R}{R_0}\right)^{\eta\beta_0}\left(\left[\frac{R}{R_0}\right]^\eta + 1\right)^{-\alpha_0-\beta_0}$$

radial basis functions: double power law models

the bi-orthogonality relation becomes

$$\delta_{nn'} \propto \int_{-1}^{1} \mathrm{d}\xi\,(1+\xi)^{\alpha}(1-\xi)^{\beta} P_n(\xi) P_{n'}(\xi)$$

with $\alpha = 2(\gamma_\infty - 1)/\eta - 1 + \alpha_0$ and $\beta = 2(1 - \gamma_0)/\eta - 1 + \beta_0$

bi-orthogonality relation of Jacobi polynomials

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}}{\mathrm{d}x}(1-x)^{\alpha+n}(1+x)^{\beta+n}$$

with $\alpha, \beta > -1$, i.e.

$$\alpha_0 > 2(1 - \gamma_\infty)/\eta \qquad\qquad \beta_0 > 2(\gamma_0 - 1)/\eta$$

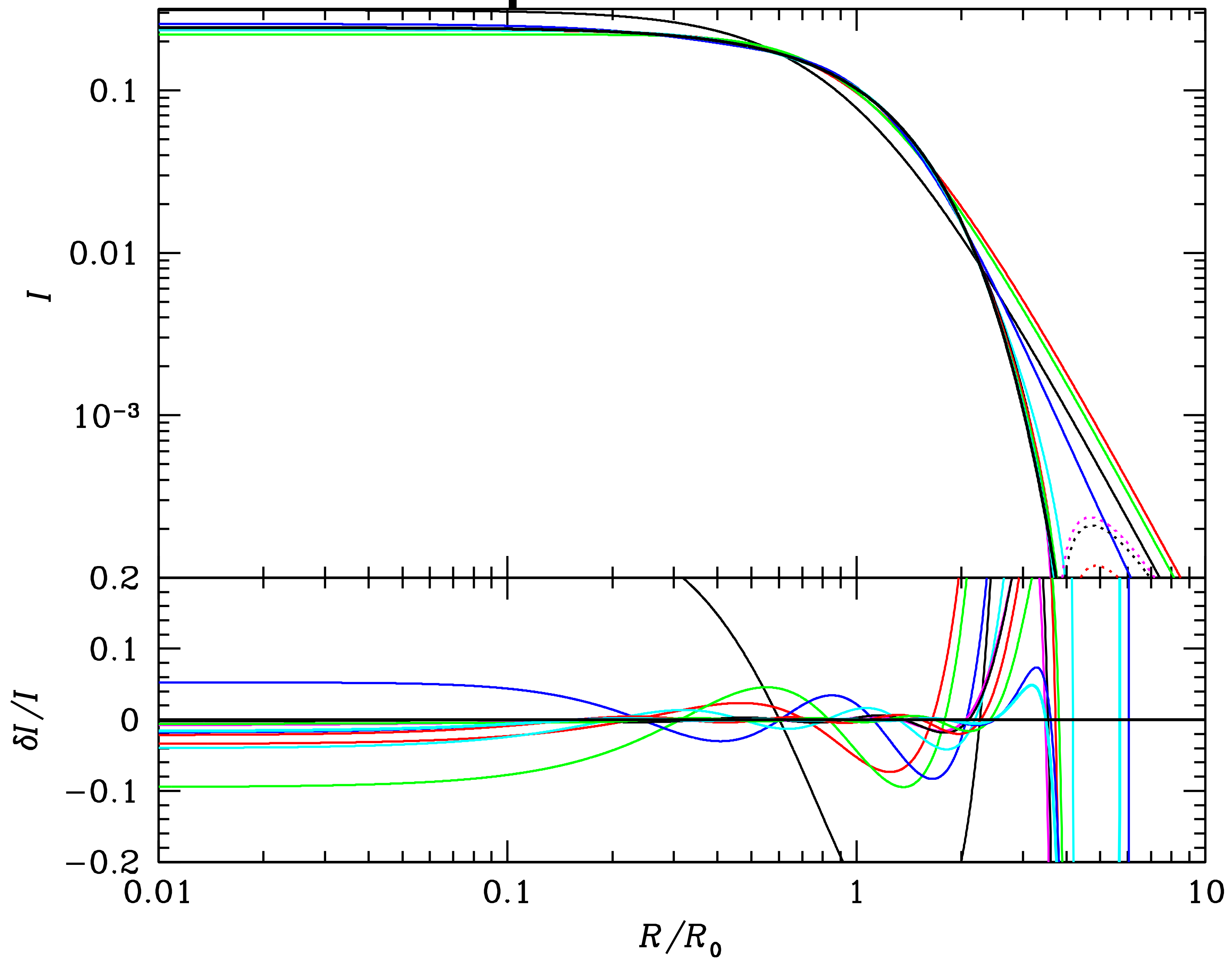
thus $S_n(R) = I_p(R)\, P_n^{\alpha,\beta}(\xi)$

=> lowest-order parameters: $R_0, \gamma_0, \gamma_\infty, \eta$

=> higher-order parameters: α_0, β_0

=> useful for power-law fall-off

radial basis functions: double power law models



fit Sersic model ($n_s = 0.6$) with double-power-law models based on Plummer ($\gamma_0 = 0, \gamma_\infty = 4, \eta = 2$) with $\alpha_0 = \beta_0 = 0$

azimuthal harmonics

complex harmonics $\exp(im\phi)$: computationally awkward

trigonometric harmonics $\cos m\phi,\ \sin m\phi$: special case $m = 0$

=> use $\mathrm{cas}(x) \equiv \cos x + \sin x$

then

$$\int_0^{2\pi} \mathrm{d}\phi\, \mathrm{cas}\, m\phi\, \mathrm{cas}\, m'\phi = 2\pi\delta_{mm'}$$

=> no complex numbers, no special cases, self-conjugate

(basis of Hartley transform: real-valued alternative to Fourier)

velocity profile: Gauß-Hermite

well-known from LOSVD modelling

$$\ell_k(v) \propto H_k(\zeta)\,\mathrm{e}^{-\zeta^2/2} \quad \text{with} \quad \zeta \equiv (v - v_0)/\sigma_0$$

error convolved basis functions

$$\tilde{\ell}_k(v_j, \sigma_j) = \int_{-\infty}^{\infty} \mathrm{d}v\,\mathrm{e}^{-\frac{(v-v_j)^2}{2\sigma_j^2}}\,\ell_k(v)$$

obey recursion relation (no need for numerics)

error de-convolution also possible analytically

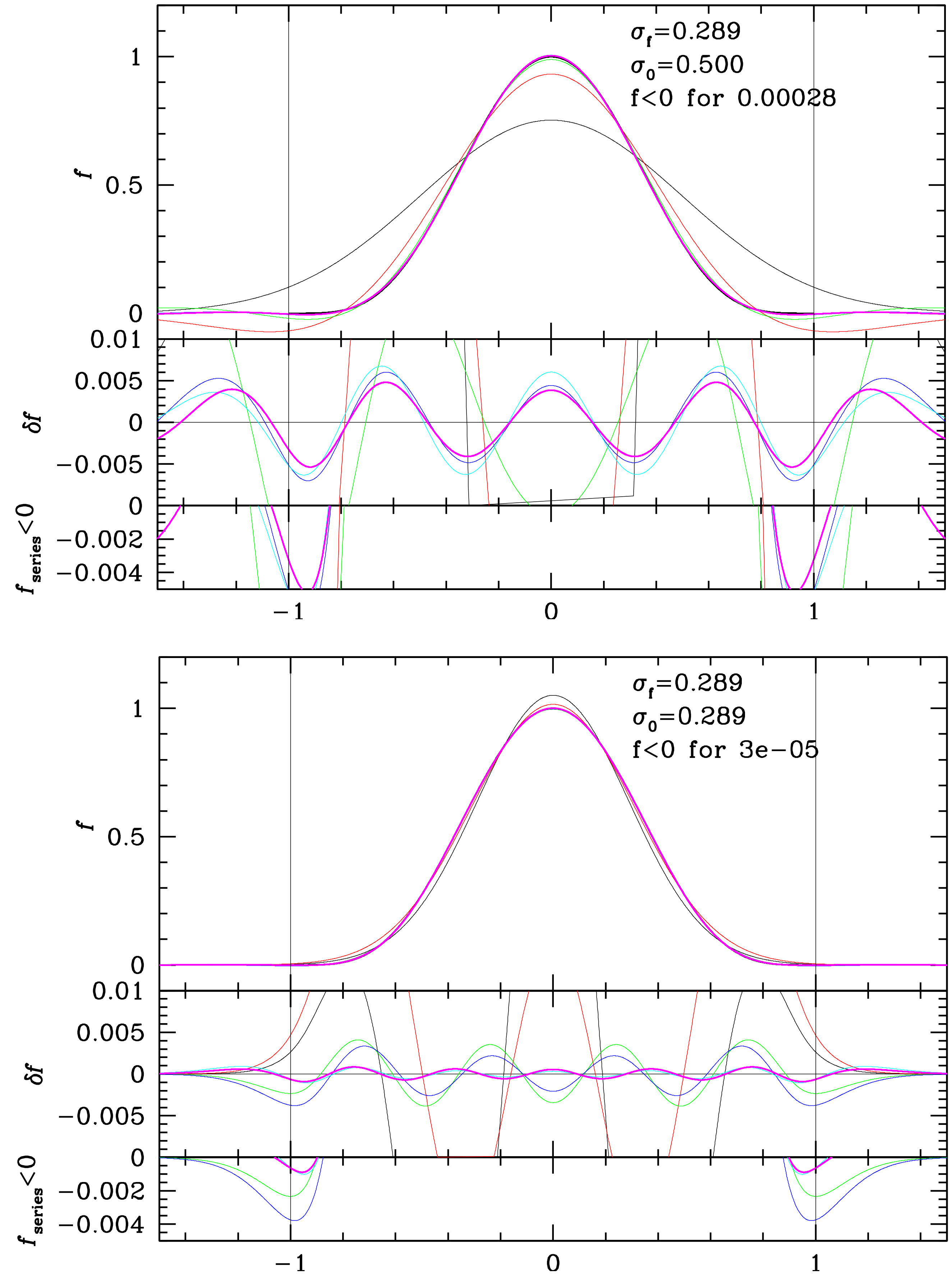
=> useful for log-likelihood & moment fitting

velocity profile: Gauß-Hermite

expanding $f(v) = (1 - v^2)^{9/2}$

expansion may obtain $f < 0$

expansion has $f \neq 0$
outside escape velocity



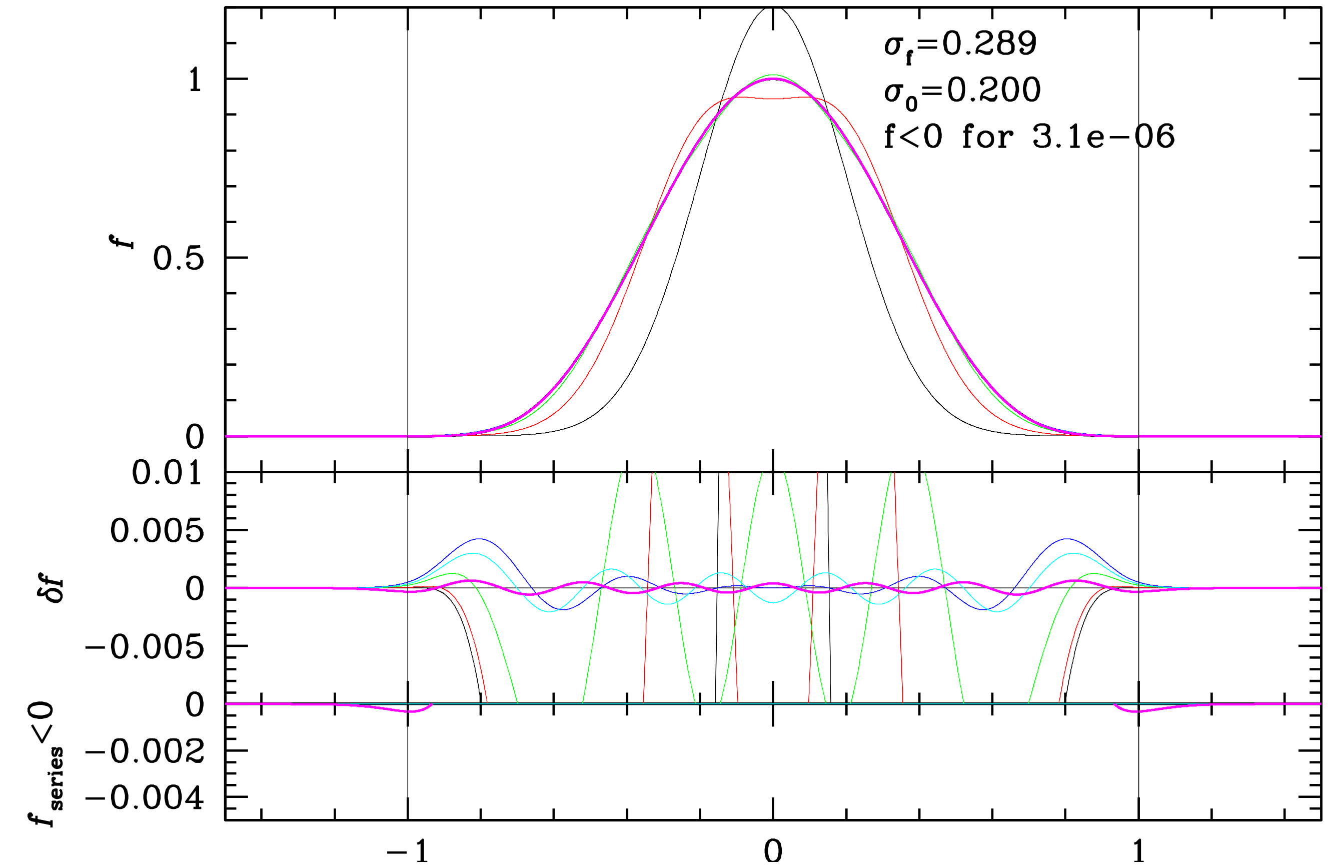
velocity profile: Gauß-Hermite

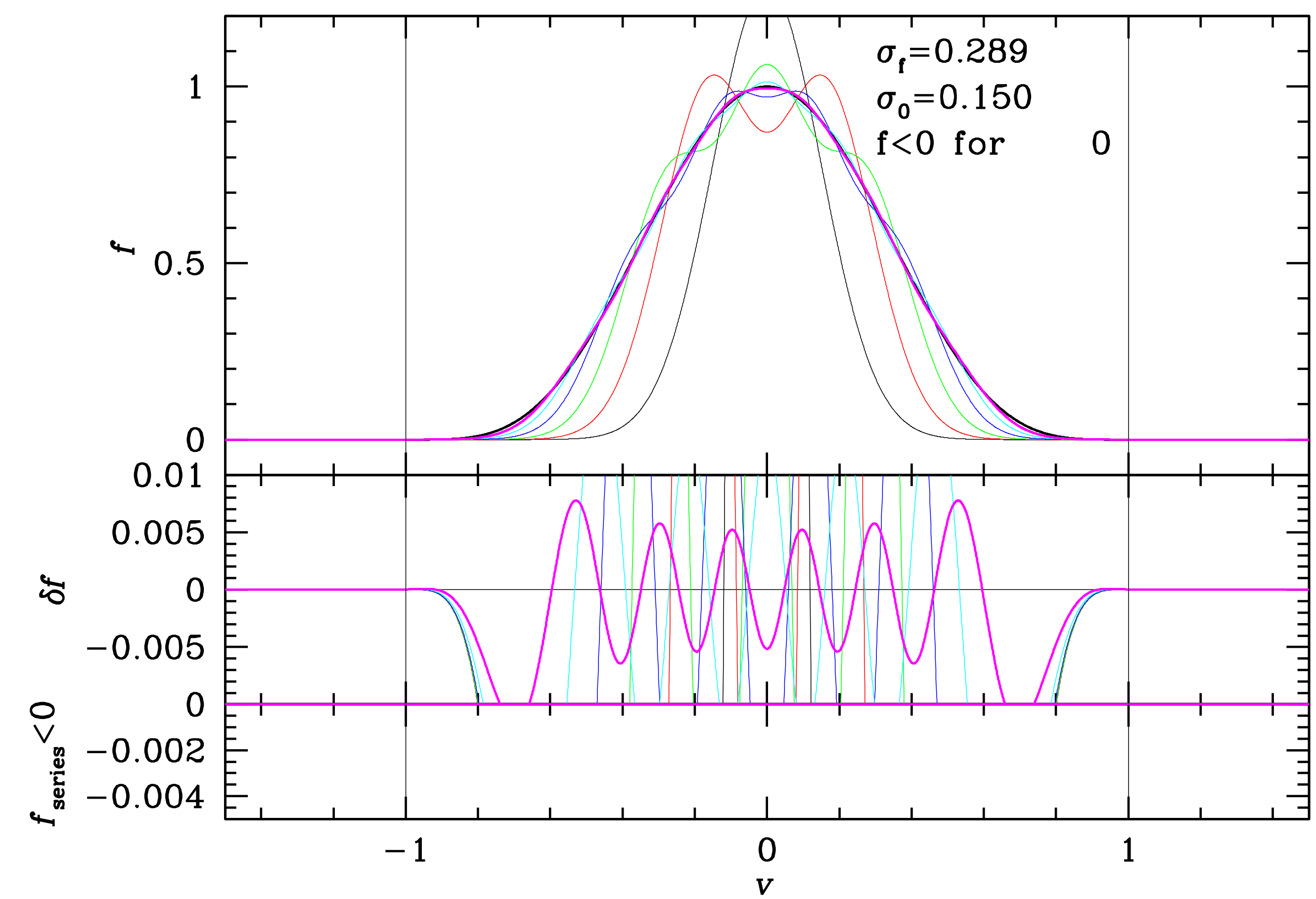
expanding $f(v) = (1 - v^2)^{9/2}$

expansion may obtain $f < 0$

expansion has $f \neq 0$
outside escape velocity

depending on parameters





velocity profile: powers

basis functions

$$\ell_k(v) \propto (1-u)^{\alpha/2}\ (1+u)^{\beta/2}\ P_k^{\alpha\beta}(u) \quad \text{for} \quad u \equiv v/v_\mathrm{e} \in [-1,1]$$

lowest order has mean & dispersion velocity

$$\bar{v} = \frac{\beta-\alpha}{\alpha+\beta+4}\, v_\mathrm{e} \quad \text{and} \quad \sigma^2 = \frac{8(\alpha+2)(\beta+2)}{(\alpha+\beta+4)^2(\alpha+\beta+6)}\, v_\mathrm{e}^2$$

which can be inverted to give

$$\alpha,\beta = \frac{(v_\mathrm{e}^2-\bar{v}^2)(v_\mathrm{e}\mp\bar{v}) - (3v_\mathrm{e}\mp\bar{v})\sigma^2}{v_\mathrm{e}\sigma^2}$$

=> parameters $v_\mathrm{e}, \bar{v}, \sigma$

=> zero outside escape velocity

=> error convolution must be done numerically

=> error de-convolution impossible

Mass profile of Fornax dSph constraints from its GCs

with Mark Wilkinson & David Cole

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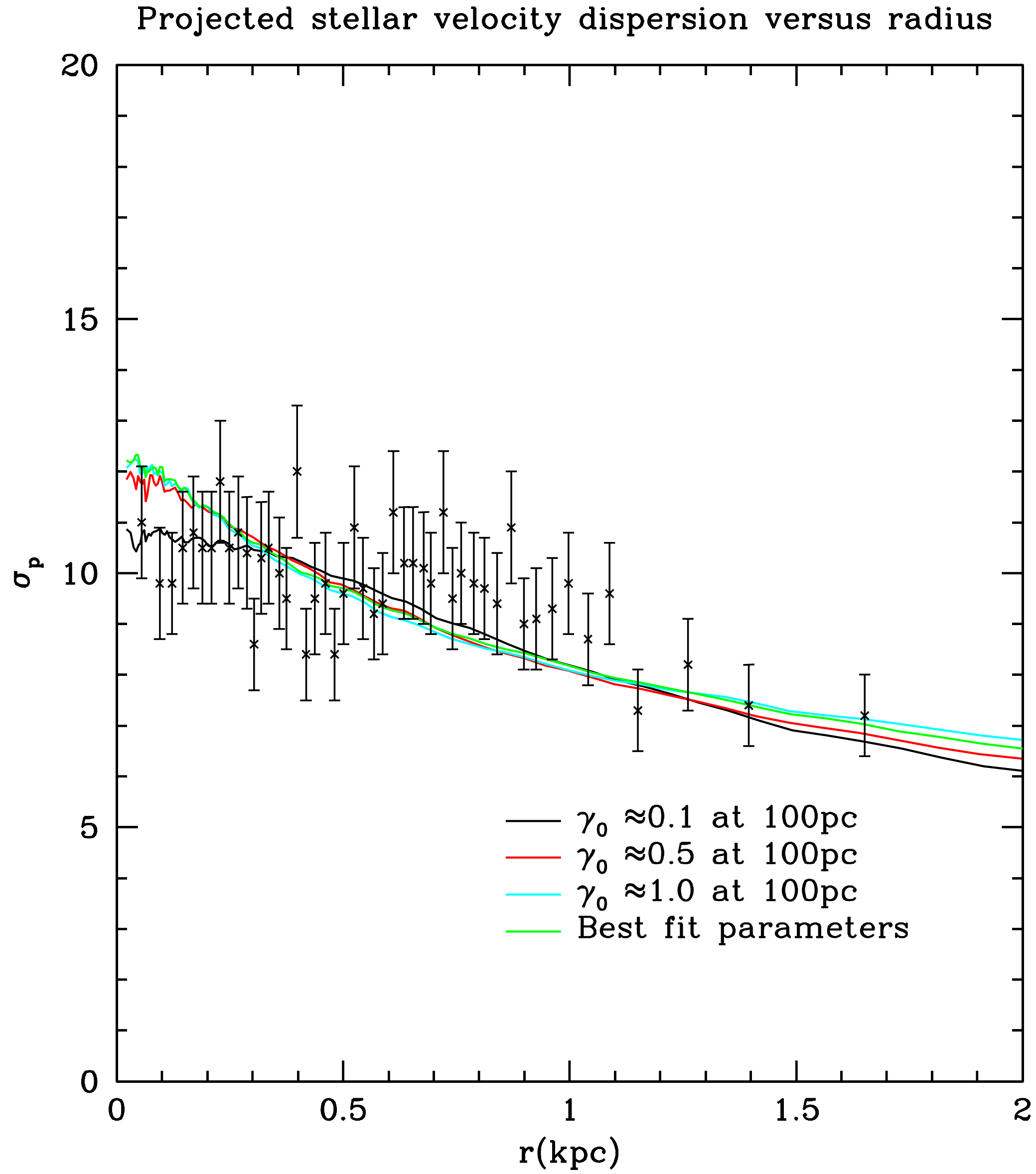
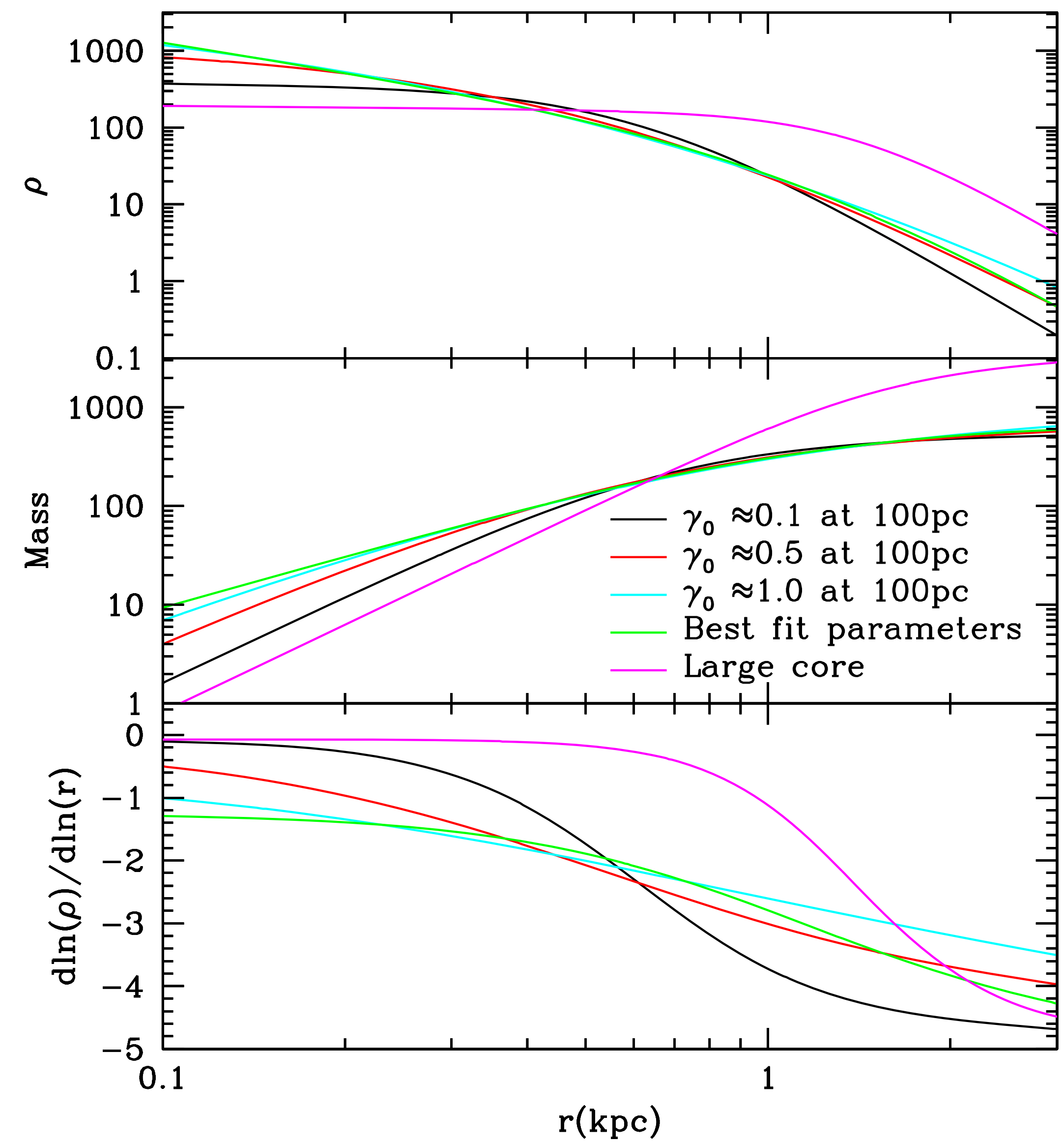
DM profile of Fornax dSph

with D. Cole, J. Read & M. Wilkinson

- modelling dynamical friction on Fornax 5 GCs
- assume 5 plausible halo models



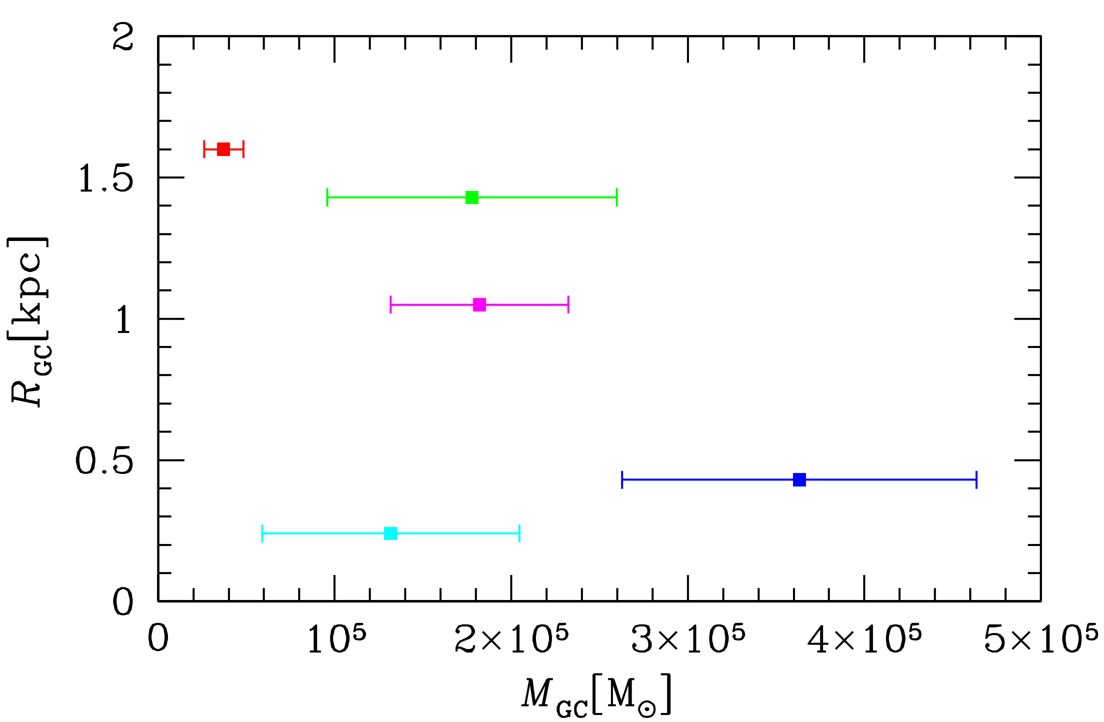


- normalise models by stellar velocity dispersion (no fit)
- cored model as suggested by Walker & Peñarrubia (2011)

DM profile of Fornax dSph

with D. Cole, J. Read & M. Wilkinson

- modelling dynamical friction on Fornax 5 GCs
- assume 5 plausible halo models
- GC consistent with stellar distribution
- lightest GC furthest away ➞ dynamical friction at work
- for each GC: assume line-of-sight position $z \in [0, 2\,\mathrm{kpc}]$
- sky velocities assuming $\beta = -0.33$ $\sigma = 10.5$ km/s
- many (700) simulations per halo model

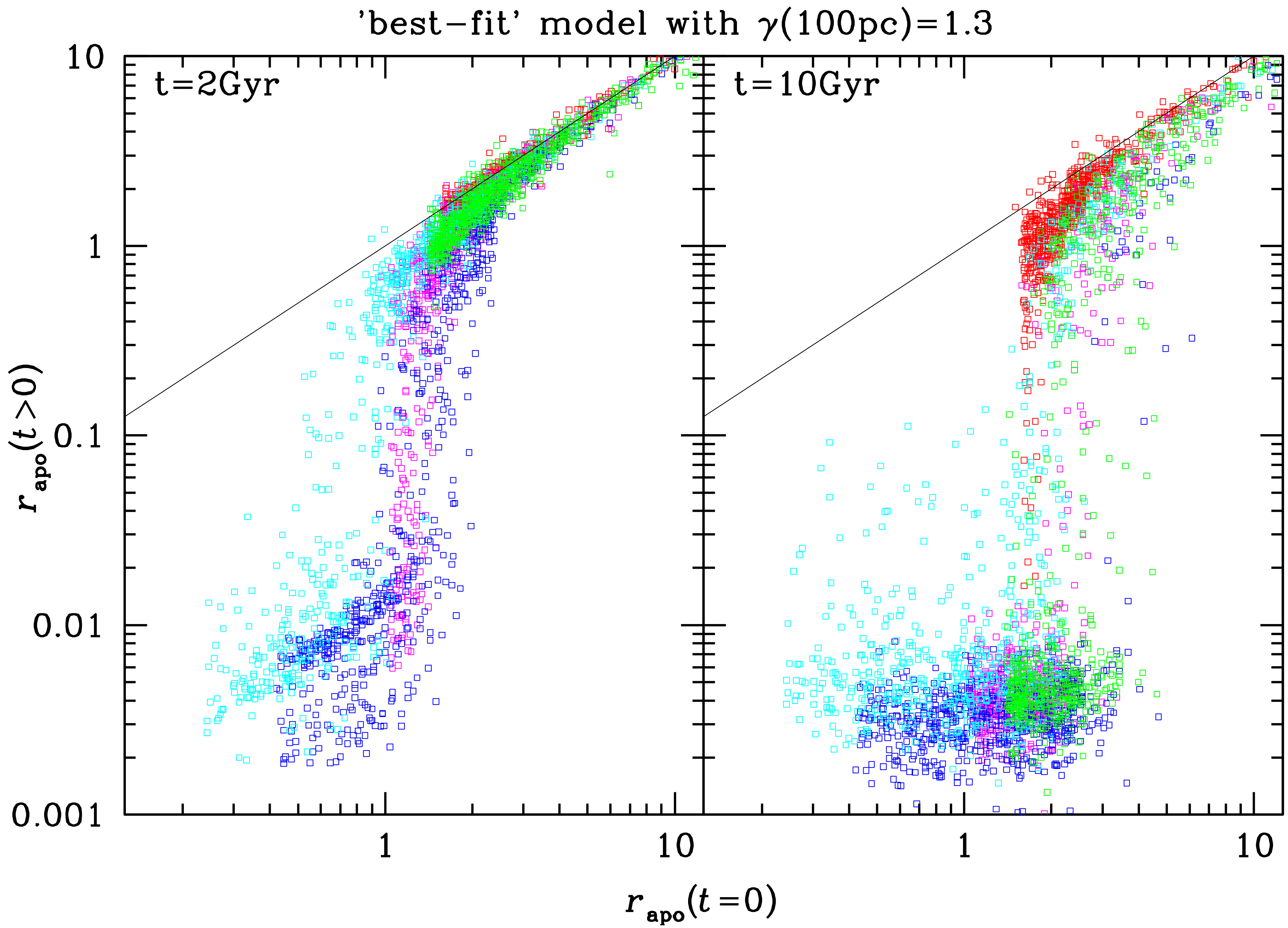


GC	log M_{GC} $(M_\odot)^a$	M_{GC} $(10^5 M_\odot)^a$	r_c^1 (pc)^a	r_p (kpc)^a	r_{los} (kpc)	v_{los} (km s^{-1})^d
Fornax	$8.15^{+0.19\,e}_{-0.37}$	1420[e]	668[e]	-	137 ± 13[b,f] 138 ± 8[g]	0
1	4.57 ±0.13	0.37	10.03	1.6	130.6 ± 3.0[b]	-
2	5.26 ±0.12	1.82	5.81	1.05	136.1 ± 3.1[b]	−1.2 ± 4.6
3	5.56 ±0.12	3.63	1.60	0.43	135.5 ± 3.1[b]	7.1 ± 3.9
4	5.12 ±0.24	1.32	1.75	0.24	134 ± 6[c]	5.9 ± 3.4
5	5.25 ±0.20	1.78	1.38	1.43	140.6 ± 3.2[b]	8.7 ± 3.6

DM profile of Fornax dSph

with D. Cole, J. Read & M. Wilkinson

- modelling dynamical friction on Fornax 5 GCs
- results for cuspy and cored halo models



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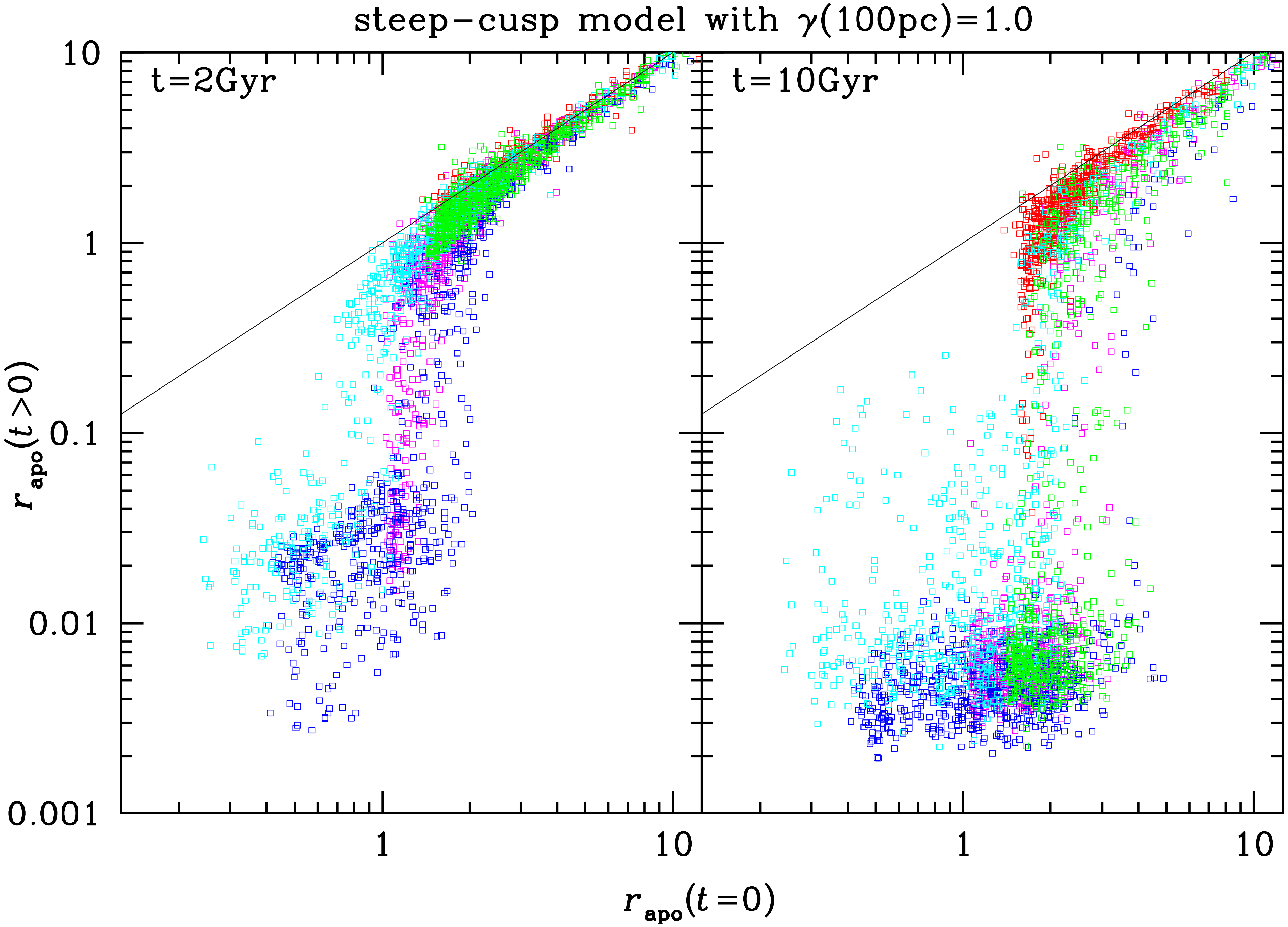

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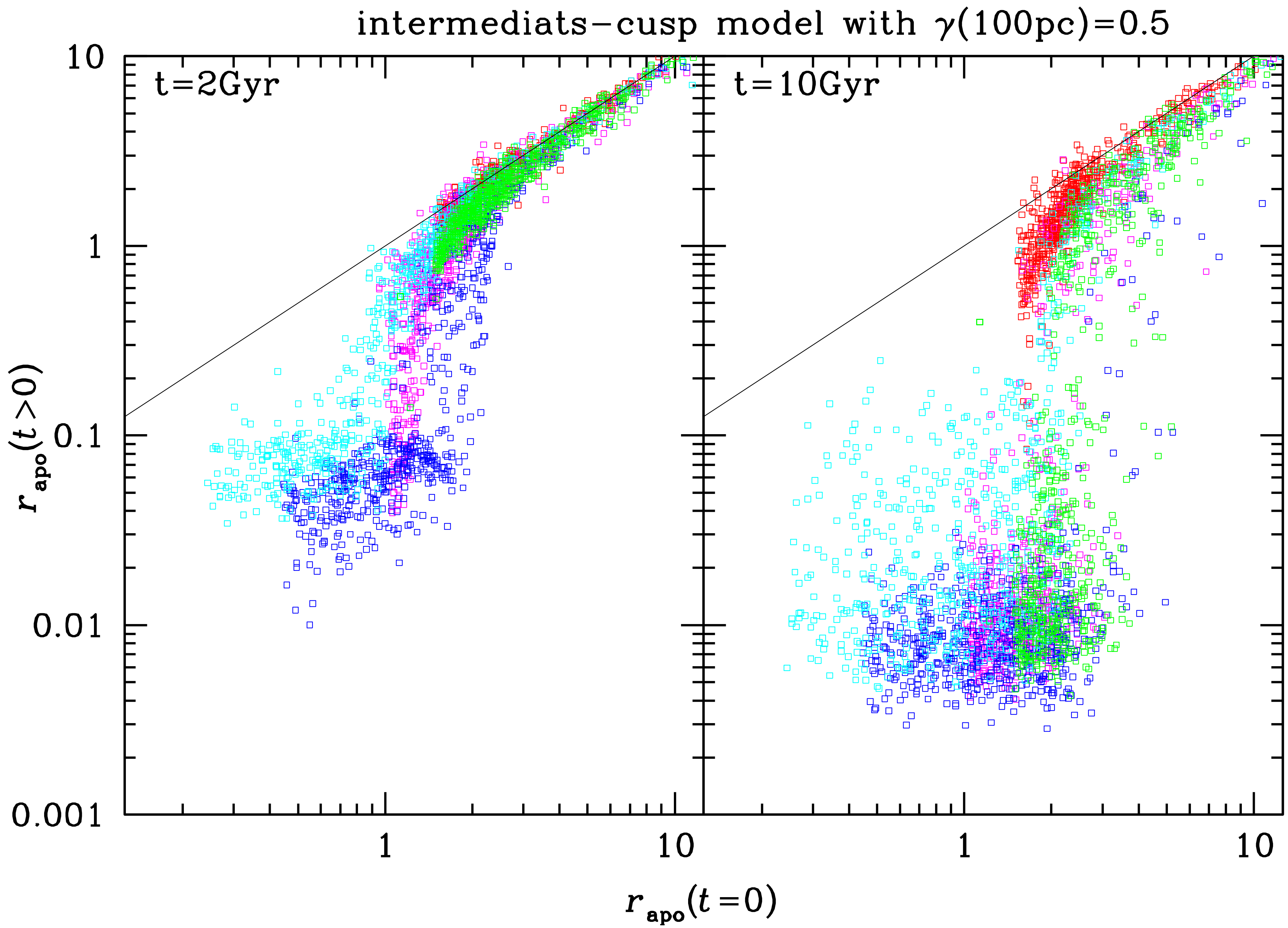

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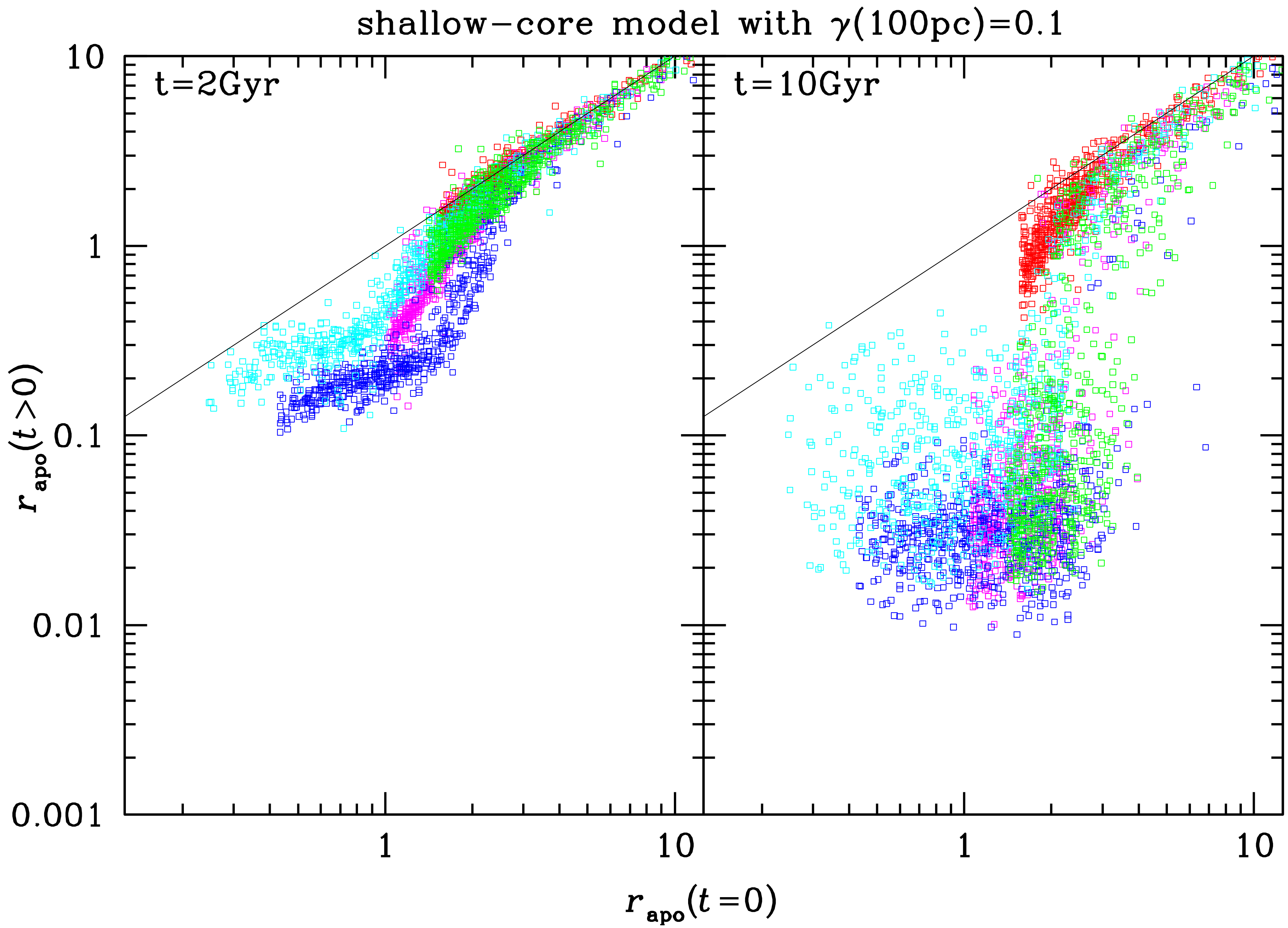

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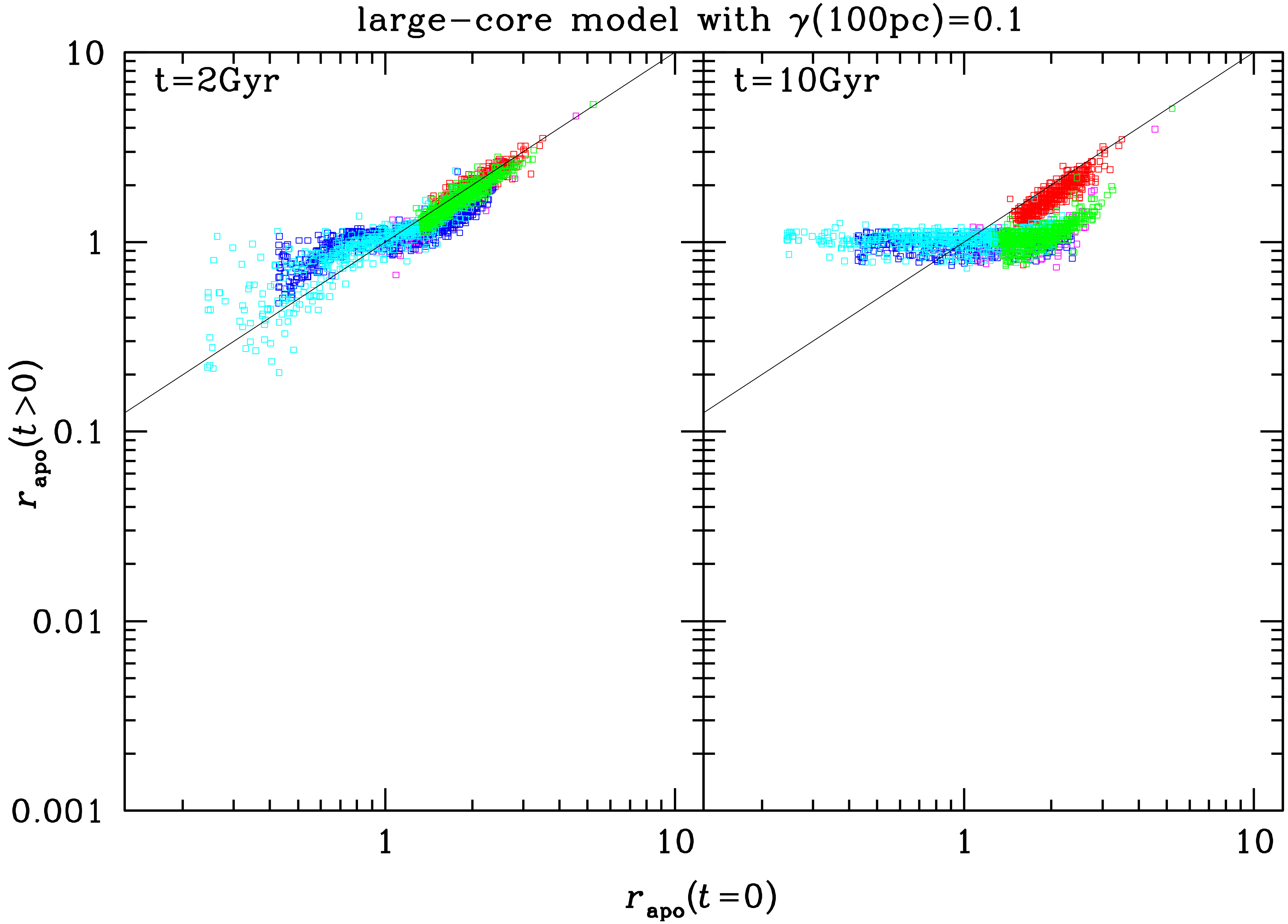

DM profile of Fornax dSph

with D. Cole, J. Read & M. Wilkinson

- modelling dynamical friction on Fornax 5 GCs
- for cuspy halo models:

  massive GCs sink into core of Fornax in $\lesssim$ 2 Gyr

  ➙ either Fornax has no cuspy halo

  ➙ or GCs have just arrived (would expect more M-segratation)

  ➙ or GCs are near peri-centre of their orbits
- for shallow cored halo models:

  in 2Gyr massive GCs sink by factor ~3-4 for most orbits

  ➙ Fornax may have a shallow core & GCs are sinking slightly

  ➙ GCs have been farther out in past
- for large-core halo (very flat inner profile):

  GCs don't sink ('dynamical buoyancy')

  ➙ GCs have settled at edge of core (would not expect M-seg.)